Renormalization group flow equations with full momentum dependence

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After a short elementary introduction to the exact renormalization group for the effective action, I discuss a particular truncation of the hierarchy of flow equations that allows for the determination of the full momentum of the n-point functions. Applications are then briefly presented, to critical $O(N)$ models, to Bose-Einstein condensation and to finite temperature field theory.

I. INTRODUCTION

The exact, or non-perturbative, renormalization group (RG) [1–5] stands out as a very promising formalism to address non-perturbative problems, i.e., problems whose solution does not appear to be expressible as an expansion in some small parameter. It leads to exact flow equations which cannot be solved in general, but which offer the possibility for new approximation schemes. It has been applied successfully to a variety of problems, in condensed matter, particle or nuclear physics (for reviews with various points of view on the subject, see e.g. [6–11]).

When only correlation functions at small momenta are needed, as is the case for instance in the calculation of critical exponents, a general approximation method to solve the infinite hierarchy of the flow equations has been developed [5, 7, 8]. This method is based on a derivative expansion of the effective action. However, in many situations, this is not enough: in order to calculate the quantities of physical interest, the knowledge of the full momentum dependence of the correlation functions is mandatory.

The present paper deals with this issue, and summarizes work that has been done following the original suggestion by Blaizot, Méndez-Galain and Wschebor (BMW) [12, 13] to obtain the momentum dependence of n-point functions from the flow equations. The strategy put forward in [12] is based on the fact that the RG flow at scale κ involves the integration of fluctuations with momenta $q \leq \kappa$, as insured by the presence of a cutoff function $R_\kappa(q)$. Since this cutoff function also guarantees that the vertex functions are smooth functions of the momenta, these can be expanded in powers of q^2/κ^2. The "leading order" (LO) of the approximation scheme consists in a truncation at the level of the flow equation for the two-point function $\Gamma^{(2)}_\kappa$, setting $q = 0$ in the vertices $\Gamma^{(3)}_\kappa$ and $\Gamma^{(4)}_\kappa$ that appear in this flow

equation. Doing so, and working in a constant external field, one can then express $\Gamma_\kappa^{(3)}$ and $\Gamma_\kappa^{(4)}$ as derivatives of $\Gamma_\kappa^{(2)}$ with respect to the background field, thereby closing the hierarchy of flow equations. Next-to-leading orders are defined by similarly truncating the hierarchy at the level of higher point functions.

The price to pay is that the flow equations become differential equations with respect to a uniform background field, with integral kernels that involve the solution itself. These non-linear integro-differential equations are a priori difficult to solve. It is possible to do so, however, with a numerical effort comparable to that involved in solving the flow equations that result from the derivative expansion. We shall provide here examples of results obtained with this method in various contexts.

The outline of the paper is as follows. In the Sect. II we briefly recall some basic features of the exact renormalization group, and derive the flow equation for the effective action. We also digress on the use of the variational principle to obtain flow equations in the context of the hamiltonian formalism often used in dealing with non relativistic systems. Then, in Sect. III, we discuss approximation schemes, such as the local potential approximation, and the BMW approximation scheme. Connections with the formalism of the two-particle irreducible effective action are also discussed. In Sect. IV, we present three applications within scalar field theory with $O(N)$ symmetry in which the full momentum dependence of the 2-point function plays an essential role. The paper ends with a short conclusion.

II. EXACT RENORMALIZATION GROUP FLOW EQUATIONS

In this section we give an elementary introduction to the exact renormalization group, focussing on the effective action. For most of the discussion (except for the digression in subsection II B) we shall restrict ourselves to the case of a scalar field theory with the classical action

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2} \left(\partial_\mu \varphi(x)\right)^2 + \frac{m^2}{2}\, \varphi^2(x) + \frac{u}{4!}\, \varphi^4(x) \right\}, \tag{1}$$

whose parameters m and u are defined at some "microscopic scale" Λ, to be specified.

A. Flow equations for the n-point functions and their generating functional

The strategy of the version of the renormalization group that we consider here is to build a family of theories indexed by a continuous parameter κ, with the dimension of a

momentum, and such that fluctuations are smoothly taken into account as κ is lowered from the microscopic scale Λ down to 0. In practice, this is achieved by adding to the original Euclidean action S a (non-local) term, quadratic in the fields, of the form

$$\Delta S_\kappa[\varphi] = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\, R_\kappa(q)\, \varphi(q)\varphi(-q). \tag{2}$$

The regulator, or cut-off function, $R_\kappa(q^2)$ is chosen so that: i) $R_\kappa(q \ll \kappa) \sim \kappa^2$ which effectively suppresses the contribution of the modes $\varphi(q \lesssim \kappa)$ by giving them a mass $\sim \kappa$; ii) it vanishes rapidly for $q \gtrsim \kappa$, leaving the modes $\varphi(q \gtrsim \kappa)$ unaffected. The explicit form of the cut-off function $R_\kappa(q)$ used in actual calculations will be specified later.

We consider then the "deformed" (non local) field theory with action $S_\kappa \equiv S + \Delta S_\kappa$, and the functional integral that yields the generating functional of Green's functions

$$Z_\kappa[j] = \int \mathcal{D}\varphi\, \mathrm{e}^{-S_\kappa + \int j\varphi}, \tag{3}$$

with $\int j\varphi \equiv \int dx j(x)\varphi(x)$. The expectation of the field in the presence of the source $j(x)$ is given by

$$\frac{\delta \ln Z_\kappa}{\delta j(x)} = \langle \varphi(x)\rangle = \phi(x). \tag{4}$$

Similarly

$$\partial_\kappa \ln Z_\kappa = -\frac{1}{2}\int_q \partial_\kappa R_\kappa(q)\langle\varphi(q)\varphi(-q)\rangle, \tag{5}$$

where we used here the fact that all the dependence on κ is contained in the regulator term $\Delta S_\kappa[\varphi]$. Note that the expectation values in Eqs. (4) and (5) are taken in the presence of the regulator, and depend therefore on κ. At this point, we define the effective action for the deformed theory, $\Gamma_\kappa[\phi]$, through a Legendre transform

$$\Gamma_\kappa[\phi] + \ln Z_\kappa[j] = \int j\phi, \qquad\qquad \frac{d\Gamma_\kappa(\phi)}{d\phi} = j. \tag{6}$$

The functional $\Gamma_\kappa[\phi]$ is the generating functional of the one-line irreducible n-point functions of the deformed theory. Taking into account that, at fixed ϕ, j depends on κ, one easily obtains the flow of $\Gamma_\kappa[\phi]$

$$\partial\Gamma_\kappa[\phi] = \frac{1}{2}\int_q \partial_\kappa R_\kappa(q)\langle\varphi(q)\varphi(-q)\rangle. \tag{7}$$

It is in fact convenient to redefine Γ_κ by subtracting from it ΔS_κ. This subtraction has the advantage to make Γ_κ coincide with the classical action S at the microscopic scale Λ (rather

than to make it coincide with $S_\Lambda = S + \Delta S_\Lambda$, as would be the case for the definition (6)). Furthermore, it modifies the flow equation in such a way that only the field fluctuations are involved. That is, once the subtraction is made, the flow is driven by the *connected* 2-point function [1]:

$$\partial_\kappa \Gamma_\kappa[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \, \partial_\kappa R_\kappa(q) \, G_\kappa(q, -q; \phi), \tag{8}$$

where

$$G(q, -q; \phi) = \langle \varphi(q)\varphi(-q) \rangle_c = \langle \varphi(q)\varphi(-q) \rangle - \phi(q)\phi(-q), \tag{9}$$

is the full propagator in the presence of the background field ϕ. Using well-known properties of the Legendre transform, one can relate $G(q, -q; \phi)$ to $\Gamma_\kappa^{(2)}[\phi]$, the second functional derivative of $\Gamma_\kappa[\phi]$ w.r.t. ϕ:

$$G_\kappa^{-1}[\phi] = \Gamma_\kappa^{(2)}[\phi] + R_\kappa. \tag{10}$$

The initial conditions on the flow equation (8) are specified at the microscopic scale $\kappa = \Lambda$ mentioned above. This scale Λ is the scale where fluctuations are damped by ΔS_κ, so that $\Gamma_{\kappa=\Lambda}[\phi] \approx S[\phi]$. It should be emphasized however that this issue of the initial condition at the scale Λ is a subtle one. It is intimately related to that of renormalization and ultraviolet divergences, issues that will not be discussed here. The effective action of the original scalar field theory is obtained as the solution of Eq. (8) for $\kappa \to 0$, at which point $R_\kappa(q^2)$ vanishes. Whether at that point the result obtained for $\Gamma[\phi]$ is independent on the choice of R_κ, that is on the path joining the classical action to the full effective action, is another subtle question, whose answer depends on the approximation made. We shall comment on it further when we discuss specific approximations.

By taking successive functional derivatives of eq. (8) with respect to ϕ, and then letting the field be constant, one gets the flow equation for the n-point functions

$$\Gamma^{(n)}(x_1, \cdots, x_n; \phi) \equiv \left. \frac{\delta^n \Gamma_\kappa}{\delta\phi(x_1) \ldots \delta\phi(x_n)} \right|_{\phi(x) \equiv \phi}, \tag{11}$$

in a constant background field. Since the background is constant, these functions are invariant under translations of the coordinates, and it is convenient to factor out of the definition of their Fourier transform the δ-function that expresses the conservation of the total momentum. Thus, with an obvious abuse of notation, we define the n-point functions $\Gamma_\kappa^{(n)}(p_1, \ldots, p_n; \phi)$

as:

$$(2\pi)^d\,\delta^{(d)}\,(p_1+\cdots+p_n)\;\Gamma_\kappa^{(n)}(p_1,\ldots,p_n;\phi) \\ \equiv \int d^dx_1\ldots\int d^dx_n\,e^{i\sum_{j=1}^n p_jx_j}\,\Gamma^{(n)}(x_1,\cdots,x_n;\phi). \tag{12}$$

We use here the convention of incoming momenta, and it is understood that in $\Gamma_\kappa^{(n)}(p_1,\ldots,p_n;\phi)$ the sum of all momenta vanishes, so that $\Gamma_\kappa^{(n)}(p_1,\ldots,p_n;\phi)$ is actually a function of $n-1$ momentum variables. We shall often use the simplified notation $\Gamma_\kappa^{(2)}(p;\phi)$ for the function $\Gamma_\kappa^{(2)}(p,-p;\phi)$.

When ϕ is constant, the functional $\Gamma_\kappa[\phi]$ itself reduces, to within a volume factor Ω to the effective potential $V_\kappa(\phi)$:

$$\Gamma_\kappa[\phi]=\Omega V_\kappa[\phi],\qquad \phi\text{ constant.} \tag{13}$$

The flow equation for the effective potential V_κ follows from that of the effective action Γ_κ, Eq. (8), when restricted to constant ϕ. It reads

$$\kappa\partial_\kappa V_\kappa(\phi)=\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\,\kappa\partial_\kappa R_\kappa(q)\,G_\kappa(q,\phi), \tag{14}$$

where

$$G_\kappa^{-1}(q,\phi)\equiv\Gamma_\kappa^{(2)}(q,\phi)+R_\kappa(q). \tag{15}$$

By taking two derivatives of eq. (8) with respect to ϕ, and then letting the field be constant, one obtains the equation for the 2-point function:

$$\partial_\kappa\Gamma_\kappa^{(2)}(p,\phi)\;=\;\int\frac{d^dq}{(2\pi)^d}\,\partial_\kappa R_\kappa(q)\,G_\kappa^2(q,\phi) \\ \times\;\left\{\Gamma_\kappa^{(3)}(p,q,-p-q;\phi)G_\kappa(q+p,\phi)\Gamma_\kappa^{(3)}(-p,p+q,-q;\phi)-\frac{1}{2}\Gamma_\kappa^{(4)}(p,-p,q,-q;\phi)\right\}. \tag{16}$$

A diagrammatic illustration of this equation is given in Fig. 1. The flow equations for the n-point functions, such as Eq. (16), constitute an infinite tower of coupled equations. The coupling between equations occurs in two ways, upwards and downwards. Upwards: typically the equation for $\Gamma^{(n)}$ involves $\Gamma^{(n+1)}$ and $\Gamma^{(n+2)}$. Downwards: all the flow equations involve $\Gamma^{(2)}$ which is coupled successively to all the equations above it. There exist also consistency conditions that we illustrate here with $\Gamma^{(2)}$ and the effective potential: $\Gamma^{(2)}(q=0)=\partial^2V/\partial\phi^2$.

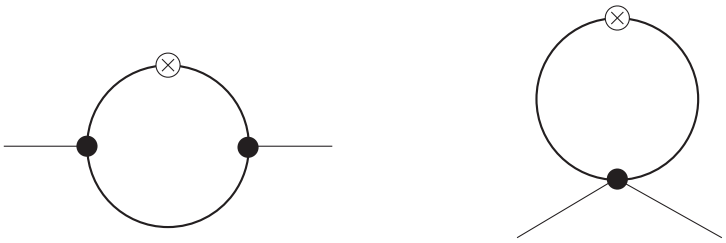

FIG. 1: The two diagrams contributing to the flow of the 2-point function, Eq. (16). The internal lines represent dressed propagators, G_κ. The cross represents an insertion of $\partial_\kappa R_k$. The vertices denoted by black dots are $\Gamma^{(3)}_\kappa$ (left) and $\Gamma^{(4)}_\kappa$ (right).

Thus, $\Gamma^{(2)}(q=0)$ can be calculated by taking the second derivative with respect to ϕ of the flow equation for the effective potential:

$$\begin{aligned}\partial_\kappa \Gamma^{(2)}_\kappa(p=0,\phi) \;&=\; \int \frac{d^d q}{(2\pi)^d}\, \partial_\kappa R_\kappa(q)\, G^2_\kappa(q,\phi) \\ &\times \left\{ G_\kappa(q,\phi) \left(\frac{\partial \Gamma^{(2)}_\kappa(q;\phi)}{\partial \phi} \right)^2 - \frac{1}{2} \frac{\partial^2 \Gamma^{(2)}_\kappa(q;\phi)}{\partial \phi^2} \right\}.\end{aligned} \tag{17}$$

Alternatively, it can be obtained from Eq. (16) where we set $p = 0$. The two calculations agree owing to the fact that

$$\Gamma^{(3)}_\kappa(q,-q,0;\phi) = \frac{\partial \Gamma^{(2)}_\kappa(q;\phi)}{\partial \phi}, \qquad \Gamma^{(4)}_\kappa(q,-q,0,0;\phi) = \frac{\partial^2 \Gamma^{(2)}_\kappa(q;\phi)}{\partial \phi^2}. \tag{18}$$

We shall consider later approximation schemes that treat at different levels of accuracy the equations for various n-point functions. This may lead to violations of such consistency conditions if one is not careful.

Relations such as (18) between n-point functions with some vanishing external momenta and derivatives of n-point functions of lower rank with respect to a constant background field play an important role in the approximation scheme presented below. In this context, we simply note here that the effective potential may be viewed as the generating functional of the n-point functions with vanishing external momenta. That is (for constant ϕ)

$$\Gamma^{(n)}_\kappa(0,\cdots,0) = \frac{\partial^n V_\kappa}{\partial \phi^n}. \tag{19}$$

In the same way, we may regard $\Gamma^{(2)}_\kappa(p;\phi)$ as the generating functional of n-point functions with one non-vanishing momentum

$$\Gamma^{(n)}_\kappa(p,-p,0,\cdots,0) = \frac{\partial^{n-2} \Gamma^{(2)}_\kappa(p;\phi)}{\partial \phi^{n-2}}. \tag{20}$$

B. Flow equation from the variational principle

The previous derivation of the flow equation did not refer to the interpretation of the various manipulations that were done in terms of "coarse graining" or "elimination of degrees of freedom", that are essential aspects of the renormalization group. These are actually hidden in the specific functional form of the regulator $R_\kappa(q)$, but nothing in the derivation of the flow equation actually depends on the choice of the regulator (asides from being quadratic in the fields, but even that can be generalized). Thus, one may view the flow equation as merely a tool to continuously go from a "simple" action (the classical action) to the full effective action, by tuning an appropriate contribution to the action. This point of view opens new perspectives and suggests that the strategy of flow equations may have a larger flexibility than is presently exploited. As an illustration, we shall here indicate how the variational principle can be used to establish flow equations in terms of hamiltonians and wave-functions for non-relativistic systems.

The strategy we shall follow is identical to that of the previous section. The system is described by a hamiltonian H, to which is added an "external field" ΔH_κ that plays the role of the regulator. We assume that $\Delta H_{\kappa=0} = 0$, and that $H_\Lambda \equiv H + \Delta H_\Lambda$ is somehow "simple", so that for instance one can determine its ground state. Then the flow equation will take us from the ground state of this simple hamiltonian H_Λ to the ground state of the hamiltonian H.

In this context, the initial condition of the flow is not restricted, as it is in most field theoretical applications to be some "classical action". Thus, it may be advantageous to depart from the standard approaches that would take H_Λ to be the hamiltonian of independent particles, or quasi-particles, and include part of the effects of the interactions into H_Λ. In fact such a strategy was recently implemented by Machado and Dupuis, in their lattice renomalization group [14], in which they separate the on-site physics form the long range correlations, only the latter being treated by the renormalization group. Much earlier, Parola and Reatto have developed a theory of liquids in which the short range part of the interaction is treated (almost) exactly with hard sphere systems, while the long range correlations are treated by a flow equation [15, 16] that actually resembles closely Eq. (8).

The flow equation for the effective action, Eq. (8), is general and could be used in the non-relativistic many-body context. However, to bring a new perspective to the discussion, we shall go through an extremely simple derivation of a flow equation that relies on the

variational principle [17]. For simplicity, we shall assume that ΔH_κ is a one body operator, i.e., a quadratic form of creation and destruction operators, that depends on the continuous parameter κ:

$$\Delta H_\kappa = \sum_{\mathbf{q}} R_\kappa(q) a^\dagger_{\mathbf{q}} a_{\mathbf{q}}, \tag{21}$$

with $a^\dagger_{\mathbf{q}}$ and $a_{\mathbf{q}}$ creation and destruction operators (of fermions or bosons). We wish to calculate the ground state energy $\hat{E}_\kappa$ of the hamiltonian $H_\kappa = H + \Delta H_\kappa$. We call $|\Psi_\kappa\rangle$ the corresponding eigenstate. In fact we do not need the exact eigenstate, it is enough that $|\Psi_\kappa\rangle$ be determined from the variational principle

$$\delta \langle \Psi_\kappa | H + \Delta H_\kappa | \Psi_\kappa \rangle |_{R_\kappa} = 0, \tag{22}$$

where it is understood that the regulator R_κ remains fixed in the variation. The flow equation for $\hat{E}_\kappa$ follows then immediately

$$\partial_\kappa \hat{E}_\kappa = \sum_{\mathbf{q}} \partial_\kappa R_\kappa(q) \, \langle \Psi_\kappa | a^\dagger_{\mathbf{q}} a_{\mathbf{q}} | \Psi_\kappa \rangle, \tag{23}$$

where $\langle \Psi_\kappa | a^\dagger_{\mathbf{q}} a_{\mathbf{q}} | \Psi_\kappa \rangle$ can be seen as an occupation factor. The equation above is the analog of Eq. (5). It just describes the change in the ground state energy under the change of the parameters of the hamiltonian. Of course, when some creation or destruction operators acquire expectation values (as in the case of Bose-Einstein condensation), it may be convenient to work with the connected part of $\langle \Psi_\kappa | a^\dagger_{\mathbf{p}} a_{\mathbf{p}} | \Psi_\kappa \rangle$, as we did earlier. Note that the flow of the energy $E_\kappa = \langle \Psi_\kappa | H | \Psi_\kappa \rangle$ reads

$$\partial_\kappa E_\kappa = - \sum_{\mathbf{q}} R_\kappa(q) \, \partial_\kappa \langle \Psi_\kappa | a^\dagger_q a_q | \Psi_\kappa \rangle. \tag{24}$$

This approach is useful if we can determine (approximately) the wave function, or at least the occupation factors, as a function of κ. It was applied in the study of the BCS-BEC crossover [17]: in this case the BCS wave function leads to a reasonably accurate estimate of the occupation factors and their dependence on κ, and this extremely simple scheme allowed us to reproduce qualitatively (and even semi quantitatively) the results of more complete renormalization group studies, such as those carried out in Ref. [18] .

III. APPROXIMATION SCHEMES

The flow equations for the n-point functions that have been presented in the previous section are exact. Their solution requires, in general, approximations. It is precisely one of

the virtues of the formulation of field theory based on the exact renormalization group to suggest approximations that are not easily derived in other, more conventional approaches. Particularly interesting are the approximation schemes for the effective action itself, that is, approximations that apply to the whole set of n-point functions at once. The approximation schemes to the discussed in this section have this property.

A. The local potential approximation

The local potential approximation (LPA) is the simplest of such approximations. It consists in assuming that for all values of κ the effective action takes the form

$$\Gamma_\kappa[\phi] = \int d^dx \left\{ \frac{1}{2} (\partial\phi)^2 + V_\kappa(\phi) \right\}. \tag{25}$$

Within this approximation, the 2-point function, obtained by differentiating $\Gamma_\kappa[\phi]$ in Eq. (25) twice with respect to ϕ, and letting ϕ be constant, is of the form ($\rho \equiv \phi^2/2$)

$$\Gamma^{(2)}_\kappa(q;\rho) = q^2 + m^2_\kappa(\rho) \qquad m^2_\kappa(\rho) \equiv \frac{\partial^2 V_\kappa}{\partial\phi^2}. \tag{26}$$

The corresponding propagator is simply a massive propagator, with a ϕ-dependent mass $m_\kappa(\rho)$. The flow equation for the effective potential V_κ becomes a closed equation that we write as follows

$$\kappa\partial_\kappa V_\kappa(\rho) = \frac{1}{2} I_1(\kappa,\rho), \tag{27}$$

where $I_1(\kappa,\rho)$ is given by

$$I_n(\kappa,\rho) \equiv J_n(p=0,\kappa,\rho), \qquad J_n(p,\kappa,\rho) \equiv \int \frac{d^dq}{(2\pi)^d}\, \kappa\partial_\kappa R_\kappa(q)\, G_\kappa(p+q) G^{n-1}_\kappa(q). \tag{28}$$

Note that the form (27) of the equation for the effective potential is general: it would yield the exact effective potential if the propagator used to calculate I_1 was the exact propagator, instead of the LPA propagator. Note also that, within the LPA, the equation for the 2-point function at vanishing external momentum, given by Eq. (17), reduces to a closed equation

$$\kappa\partial_\kappa \Gamma^{(2)}_\kappa(0;\rho) = I_3(\kappa,\rho) \left(\frac{\partial\Gamma^{(2)}_\kappa(0;\rho)}{\partial\phi} \right)^2 - \frac{1}{2} I_2(\kappa,\rho)\, \frac{\partial^2\Gamma^{(2)}_\kappa(0;\rho)}{\partial\phi^2}, \tag{29}$$

since the derivatives of $\Gamma^{(2)}_\kappa(q;\rho)$ with respect to ϕ are identical to derivatives of the effective potential, or, equivalently, to derivatives of $\Gamma^{(2)}_\kappa(0;\rho)$.

The LPA has been widely used, and given its simplicity, the quality of the results obtained is quite good [8, 21–23]. Note that the LPA, by construction, yields n-point functions that have no intrinsic dependence on the external momenta: the generating functional of these n-point functions is given by Eq. (25) taken at $\kappa = 0$, so that, for $n > 2$, the LPA n-point functions are just derivatives of the effective potential (see Eq. (19)). The LPA can be improved through an expansion in gradients of the fields, usually referred to as the derivative expansion (DE) [6, 8, 20]. Within the derivative expansion, the lowest n-point functions have polynomial dependence on the momenta (the degree of the polynomial corresponding to the order of the expansion in the derivatives). It follows in particular that the derivative expansion does not directly describe the anomalous behavior of n-point functions at small momenta (anomalous dimensions can be recovered for instance from the scale dependence of the field normalization). In order to capture the full momentum dependence of the n-point function, a better truncation scheme is necessary: the BMW scheme [12], to be discussed next, achieves this goal.

B. The approximation BMW-LO

The BMW approximation scheme relies on two observations. First, the presence of the cut-off function $R_\kappa(q)$ insures that the n-point functions $\Gamma_\kappa^{(n)}(p_i)$ remain regular functions of the external momenta p_i as $p_i \to 0$; besides, it limits the internal momentum q in equations such as Eq. (16) to $q \lesssim \kappa$. In line with this observation, the approximation consists in neglecting the q-dependence of the vertex functions in the r.h.s. of the flow equations (e.g. set $q = 0$ in $\Gamma^{(3)}$ and $\Gamma^{(4)}$ in Eq. (16)), while keeping the full dependence on the external momenta p_i. The second observation is that, for uniform fields, $\Gamma_k^{(m+1)}(p_1, \ldots, p_m, 0, \phi) = \partial_\phi \Gamma_k^{(m)}(p_1, \ldots, p_m, \phi)$ (a relation that we have already used earlier, see Eq. (18)). This enables one to close the hierarchy of equations at some finite order. The order m of the scheme consists in keeping the full momentum dependence of all the n-point functions up to $\Gamma_\kappa^{(m)}$, and expressing $\Gamma_\kappa^{(m+1)}$ and $\Gamma_\kappa^{(m+2)}$ as derivatives of $\Gamma_\kappa^{(m)}$ with respect to ϕ, after setting to zero the loop momenta that flows through $\Gamma_\kappa^{(m+1)}$ and $\Gamma_\kappa^{(m+2)}$ in the equation for $\Gamma_\kappa^{(m)}$. The accuracy of the scheme depends of course of the rank m at which one operates the truncation, but obviously the implementation becomes increasingly complicated as m grows. It is therefore gratifying that accurate results can be obtained with lowest order truncations.

The approximations obtained by truncating the hierarchy at the lowest level, i.e., $m = 0$

is identical to the local potential approximation discussed in the previous subsection. The next order of the approximation scheme, which we shall in fact refer to as the leading order (LO) of the BMW method, consists in a truncation at the level $m = 2$, i.e., at the level of the flow equation of the 2-point function. The resulting equation is the closed equation

$$\kappa\partial_\kappa\Gamma_\kappa^{(2)}(p,\rho) = J_3(p,\kappa,\rho)\ \left(\frac{\partial\Gamma_\kappa^{(2)}(p,\rho)}{\partial\phi}\right)^2 - \frac{1}{2}I_2(\kappa,\rho)\ \frac{\partial^2\Gamma_\kappa^{(2)}(p,\rho)}{\partial\phi^2}, \tag{30}$$

where $J_3(p,\kappa,\rho)$ and $I_2(\kappa,\rho)$ are obtained from the general definitions in Eq. (28). This equation may be viewed as a generalization of Eq. (29) that takes into account the full momentum dependence of the 2-point function. What the BMW approximation achieves is the possibility to factorize the vertices and take them out of the integrals. The momentum dependence that remains within the 3 and 4-point vertices is that of the 2-point function itself (see Eq. (20)). Interestingly, a very similar set of equations have been obtained much earlier in the context of the theory of liquids [15], but these were unknown to the authors of Ref. [12].

At this point we recall the consistency condition discussed in connection with Eq. (17): the flow equation for $\Gamma_\kappa^{(2)}(p=0,\rho)$ can be obtained by taking the second derivative of the flow equation for the effective potential with respect to the background field. The resulting equation, Eq. (17), does not coincide here with Eq. (30) in which we set $p = 0$. This is because Eq. (30) results from the BMW truncation, which is not implemented in Eq. (17), and needs not be. This results in a mismatch between the two ways of calculating $\Gamma^{(2)}(p = 0)$, whose origin can be traced back to the fact that the 2-point function and the effective potential are not determined with the same accuracy: loosely speaking, the 2-point function calculated from Eq. (30) is accurate to order one-loop, while the effective potential obtained from Eq. (27), with the BMW propagator, is accurate to order two-loop. In order to properly deal with this feature, we treat separately the zero momentum ($p = 0$) and the non-zero momentum ($p \neq 0$) sectors, and write

$$\Gamma_\kappa^{(2)}(p,\rho) \equiv p^2 + \Delta_\kappa(p,\rho) + m_\kappa^2(\rho), \tag{31}$$

where $\Delta_\kappa(p=0,\rho) = 0$. Now, $m_\kappa^2(\rho)$ is obtained by solving the equation for the effective potential, while the equation for $\Delta_\kappa(p,\rho)$ can be easily deduced from that for $\Gamma_\kappa^{(2)}$, i.e., from Eq. (30) by subtracting the corresponding equation that holds for $p = 0$. It reads

$$\begin{aligned}\partial_t\Delta_\kappa(p,\rho) \ &= \ 2\rho J_3(p,\kappa,\rho)\ \left[u_\kappa(\rho) + \Delta'_\kappa(p,\rho)\right]^2 - 2\rho I_3(\kappa,\rho)\ u_\kappa^2(\rho) \\ &- \ \frac{1}{2}I_2(\kappa,\rho)\ \left[\Delta'_\kappa(p,\rho) + 2\rho\Delta''_\kappa(p,\rho)\right],\end{aligned} \tag{32}$$

where the symbol $'$ denotes the derivative with respect to ρ, and $u_\kappa(\rho) \equiv dm_\kappa^2(\rho)/d\rho$.

This equation (32), together with that for the effective potential, Eq. (27), and that for the propagator

$$G_\kappa^{-1}(q,\rho) = q^2 + \Delta_\kappa(q,\rho) + m_\kappa^2(\rho) + R_\kappa(q), \tag{33}$$

constitute a closed system of equations for $\Gamma_\kappa^{(2)}(p,\rho)$ which can be solved with the initial condition $\Gamma_\Lambda^{(2)}(p,\rho) = p^2 + m^2 + u\rho$.

C. Relation to the 2PI formalism

Because the exact RG formalism that we are using puts emphasis on the propagator, it is natural to ask for the connection with the 2PI (2-particle irreducible) formalism [24–26]. Let us recall that the central quantity in this formalism is $\Phi[G]$, the sum of the two-particle-irreducible "skeleton" diagrams, a functional of the full propagator G. From $\Phi[G]$ one obtains the self-energy by functional differentiation (to within factors $(2\pi)^3$):

$$\Sigma(p) = 2\frac{\delta\Phi}{\delta G(p)}. \tag{34}$$

This relation, together with Dyson's equation:

$$G^{-1}(p) = p^2 + m^2 + \Sigma(p)\,, \tag{35}$$

defines the physical propagator and self-energy in a self-consistent way. We shall refer to Eq. (35), with $\Sigma[G]$ given by Eq. (34), as the "gap equation". A further differentiation of $\Phi[G]$ with respect to G yields the two-particle-irreducible kernel

$$\mathcal{I}(q,p) = 2\frac{\delta\Sigma(p)}{\delta G(q)} = 4\frac{\delta^2\Phi}{\delta G(q)\delta G(p)} = \mathcal{I}(p,q), \tag{36}$$

of a Bethe-Salpeter type equation

$$\Gamma^{(4)}(q,p) \;=\; \mathcal{I}(q,p) - \frac{1}{2}\int_l \Gamma^{(4)}(q,l)\,G^2(l)\,\mathcal{I}(l,p), \tag{37}$$

that allows the calculation of the four-point function $\Gamma^{(4)}(q,p) \equiv \Gamma^{(4)}(q,-q,p,-p)$: the quantity $\mathcal{I}(q,p)$ is the two-particle-irreducible contribution to $\Gamma^{(4)}(q,p)$ in one particular channel. If all skeletons are kept in Φ, these relations are exact. A Φ-derivable approximation [25] is obtained by selecting a class of skeletons in Φ and calculating Σ and $\Gamma^{(4)}$ from the equations above.

The 2PI formalism provides a set of functional relations among the n-point functions that can be used to define a truncation of the flow equations [27]. Consider indeed the equation for the 2-point function for a vanishing background field. It reads:

$$\partial_\kappa \Gamma^{(2)}_\kappa(p) = \; - \; \frac{1}{2} \int_q \partial_\kappa R_\kappa(q) \, G^2_\kappa(q) \, \Gamma^{(4)}_\kappa(q,p). \tag{38}$$

A possible truncation consists in using for $\Gamma^{(4)}_\kappa(q,p)$ in the right-hand-side of this equation, the extension of the relation (37) to the deformed theory, namely

$$\Gamma^{(4)}_\kappa(q,p) = \mathcal{I}_\kappa(q,p) \; - \; \frac{1}{2} \int_l \Gamma^{(4)}_\kappa(q,l) \, G^2_\kappa(l) \, \mathcal{I}_\kappa(l,p) \, , \tag{39}$$

where the subscript κ on $\mathcal{I}_\kappa$ means that the functional derivative defining the kernel $\mathcal{I}$ (see Eq. (36)) is to be evaluated for $G = G_\kappa$, with $G_\kappa^{-1} = \Gamma^{(2)}_\kappa + R_\kappa$. Since $\mathcal{I}_\kappa$ is a functional of the 2-point function, the system of equations (38-39) is indeed closed.

One nice feature of this truncation scheme is that it is systematically improvable, by adding more skeletons to Φ: if all skeletons are included, the solution of the coupled system of equations (38-39) provides the exact 2-point function as well as the exact 4-point function for a particular configuration of the external momenta. A second attractive feature is that it preserves the property of the flow of being a total derivative with respect to the parameter κ. It is indeed not difficult to show, using Eqs. (39) and (38), that

$$\partial_\kappa \Gamma^{(2)}_\kappa(p) = \partial_\kappa \Sigma_\kappa(p), \tag{40}$$

where $\Sigma_\kappa \equiv \Sigma[G_\kappa]$, with $\Sigma[G]$ given by Eq. (34). This is a unique property of this truncation, that is not shared by most other popular truncations of the exact RG (with the noticeable exceptions of the perturbative expansion, and the large N approximation, see e.g. [33]. A similar property of the flow equation was also obtained in the off-equilibrium context in Ref. [30]. The BMW truncation does not respect this property). Since it is a total derivative, the flow can be easily integrated to yield the gap equation whose solution is equivalent to a resummation of all the Feynman diagrams that are generated from the skeletons that are kept in the approximation considered. And because the solution of the gap equation corresponds to an exact resummation of selected Feynman diagrams, at the end of the flow where $\kappa = 0$ and the regulator vanishes, the final result is rigorously independent of the choice of the regulator.

All these properties of the 2PI truncation may look at first disappointing from the point of view of the flow equations: indeed, all what the flow does in this particular truncation is

solving the 2PI equations! However, there is certainly interest in establishing direct connections between non trivial non-perturbative approximations. In particular, because the 2PI truncations lead to flow equations that are exact derivatives, they could be used to test other approximations. Besides, from the point of view of the 2PI formalism, there is a practical advantage in reformulating the gap equation as a flow equation: this is because initial value problems are in general easier to solve than non linear gap equations. Furthermore, regarding the 2PI equations as flow equations shed a new light on the renormalization of Φ-derivable approximations [28, 29].

IV. APPLICATIONS

We turn now to a few specific applications of the non perturbative renormalization group using the BMW truncation scheme. Most of these applications concern scalar field theories with an $O(N)$ symmetric action of the generic form

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2} \left(\partial_\mu \boldsymbol{\varphi}(x)\right)^2 + \frac{m^2}{2} \boldsymbol{\varphi}^2(x) + \frac{u}{4!} \left(\boldsymbol{\varphi}^2(x)\right)^2 \right\}, \qquad \boldsymbol{\varphi}^2 = \sum_{a=1}^{N} \varphi_a \varphi_a. \tag{41}$$

The first application covered in this section may be seen as a "classic" application within the context of the renormalization group: it concerns the critical $O(N)$ models, and the calculation of critical exponents and the scaling functions [31, 32]. The second application refers to the shift of the transition temperature of the Bose-Einstein condensation in a dilute Bose gas: this again involve the $O(N)$ field theory (with $N = 2$), but the relevant quantity to be calculated is sensitive to the whole momentum range of the 2-point function, and not only to the critical momentum region [33]. The last application concerns the thermodynamics of quantum fields. It shows how the exact renormalization group allows one to circumvent the specific difficulties of perturbation theory in such systems. The physics motivation is the physics of the quark-gluon plasma, but some of the essential difficulties of perturbation theory in QCD at finite temperature are well illustrated again by the scalar field [34, 35].

Before turning to these applications, we need to comment on the dependence of the regulator in the practical calculations. We shall use in our calculations a cut-off function of the generic form

$$R_\kappa(q) = Z_\kappa \kappa^2 r(\tilde{q}), \qquad \tilde{q} \equiv \frac{q}{\kappa}, \tag{42}$$

where Z_κ is a function of κ only. This factor Z_κ reflects the finite change in normalization of

the field between the scale Λ and the scale κ. It can be defined as

$$Z_\kappa = \left. \frac{\partial \Gamma_\kappa^{(2)}(p;\rho)}{\partial p^2} \right|_{p_0;\rho_0}, \tag{43}$$

with ρ_0 and p_0 a priori arbitrary. In practice, one usually chooses $p_0 = 0$ and ρ_0 the value of ρ at the minimum of the effective potential. The factor Z_κ enters the scaling to dimensionless variables used for the numerical solution in the critical region.

The results of the calculations presented below were obtained with an exponential regulator

$$r(\tilde{q}) = \frac{\alpha \tilde{q}^2}{\mathrm{e}^{\tilde{q}^2} - 1}, \tag{44}$$

where α is a free parameter. As we shall see, physical quantities exhibit a small dependence on α. Since in the absence of any approximation, they would be strictly independent of the cut-off function, a study of this spurious dependence provides an indication of the quality of the approximation [9, 67].

A. Critical $O(N)$ models

The equations of the BMW method have been solved first with additional approximations in Refs. [19], but the results presented here were obtained by solving numerically the nonlinear integro-partial-differential equations (32) and (27) without any further approximation [31]. The numerical techniques used are described in Refs. [31] and [32]. Here, we report only some selected results, and comment on important aspects of the approximation that are needed in order to gauge the quality of these results.

Table I contains results for the critical exponents η and ν, in dimension $d = 3$ and for various values of N, together with some of the best estimates available in the literature. To appreciate the quality of these results, let us recall that these depend a priori on the parameters α, p_0, and ρ_0 (see Eqs. (42) and (43)). (They also depend on the functional form of the cut-off function, but no systematic exploration has been performed to study this dependence.) In all cases studied, we find the dependence on p_0 and ρ_0 to be much smaller than that on α, so that only the latter needs to be considered. As a function of α, physical quantities typically exhibit a single extremum α^* located near $\alpha = 2$. Moreover, we find this extremum generally pointing towards the best numerical estimates. Since, in the absence of any approximation there should be no dependence on α, we regard these extremum values,

TABLE I: Critical exponents for the $O(N)$ models for $d = 3$.

N	BMW		Resummed pert. exp.			MC and high-temp. series		
	η	ν	η	ν	Ref.	η	ν	Ref.
0	0.034	0.589	0.0284(25)	0.5882(11)	[36]	0.030(3)	0.5872(5)	[38]
1	0.039	0.632	0.0335(25)	0.6304(13)	[36]	0.0368(2)	0.6302(1)	[37]
2	0.041	0.674	0.0354(25)	0.6703(15)	[36]	0.0381(2)	0.6717(1)	[39]
3	0.040	0.715	0.0355(25)	0.7073(35)	[36]	0.0375(5)	0.7112(5)	[40][41]
4	0.038	0.754	0.035(4)	0.741(6)	[36]	0.0365(10)	0.749(2)	[41]
10	0.022	0.889	0.024	0.859	[42]			

being locally independent of α, as our best values, adopting in doing so a strategy often referred to as the principle of minimal sensitivity (PMS) [44]. Our numbers are then all given for the PMS values α^* of the regulator parameter, and the digits quoted are those which remain stable when α varies in the range $[\alpha^* - \frac{1}{2}, \alpha^* + \frac{1}{2}]$. The quality of the values obtained for the critical exponents is obvious: For all N the results for ν agree with the best estimates to within less than a percent; as for the values of η it is typically at the same distance from the Monte-Carlo and temperature series estimates as resummed perturbative data. For $N = 100$, we find $\eta = 0.0023$, and $\nu = 0.990$, which compare well to the values $\eta = 0.0027$ and $\nu = 0.989$ obtained in the $1/N$ expansion [45]. Our numbers also compare favorably with those obtained at order ∂^2 in the DE scheme[20].

The two-dimensional case, for which exact results exist, provides an even more stringent test of the BMW scheme. We have results at the moment only for the Ising model, i.e., for $N = 1$, which exhibits a standard critical behavior in $d = 2$. The perturbative method that works well in $d = 3$ fails here: for instance, the fixed-dimension expansion that provides the best results in $d = 3$, yields, in $d = 2$ and at five loops, $\eta = 0.145(14)$ [46], in contradiction with the exact value $\eta = \frac{1}{4}$. We find instead $\eta = 0.254$, $\nu = 1.00$. Note however that no systematic study of the dependence of this result on the regulator parameter has been performed yet.

The BMW scheme yields the complete momentum dependence of the 2-point function. All expected features of $\Gamma_k^{(2)}(p)$ at criticality are observed: In the infrared (IR) regime $\kappa \ll p \ll u$, $\Gamma_\kappa^{(2)}(p, 0) \sim p^{2-\eta}\kappa^\eta$, and this IR behavior of $\Gamma_\kappa^{(2)}(p)$ can be used to extract the value of η; the value obtained directly from the momentum dependence of $\Gamma_\kappa^{(2)}(p)$ is in excellent agreement with that deduced from the κ-dependence of the field normalization factor Z_κ.

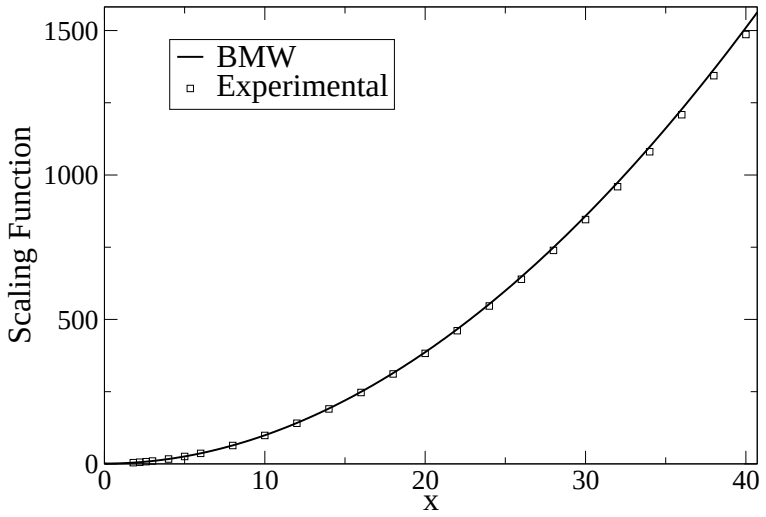

FIG. 2: The scaling function $g_+(x)$ as a function of $x = q\xi$. The experimental data are from Ref. [47].

The ultraviolet (UV) regime $\kappa, u \ll p \ll \Lambda$ exists if u is sufficiently small; this regime can be studied perturbatively and one finds that, in leading order, $\Gamma^{(2)}_\kappa(p,0)) \sim u^2 \log(p/u)$. The present approximation reproduces this logarithmic behavior with, however, a prefactor 8% larger than the two-loop result. Note that the complete two-loop behavior can be recovered by a simple improvement of the BMW scheme [32].

The momentum dependence of the 2-point function can be further tested by analyzing the so-called scaling functions

$$G_\pm(q) = \chi g_\pm(q\xi), \tag{45}$$

where $\chi^{-1} \equiv \Gamma^{(2)}(q=0)$, and ξ is the correlation length, which diverges close to criticality with the ν critical exponent. Here $\pm$ refers to the two phases, above and below the critical point, respectively. The functions $g_\pm$ are universal. We consider here the case $N = 1$. The scaling function $g_+(x = q\xi)$ has been calculated in the BMW approximation for different values of the correlation length. It is plotted in Fig. 2, where we can see that the scaling is perfectly reproduced. The agreement with the experimental data is also excellent, although this is somewhat delusive since, in this range of momenta, the physics is dominated by mean field effects, and the scaling function deviates only slightly from that predicted by the Ornstein-Zernicke approximation). A critical study is presented in Ref. [32].

B. Temperature of Bose-Einstein condensation

A quantity particularly sensitive to the UV-IR crossover region is the shift, due to interactions, of the critical temperature of the dilute, weakly interacting, Bose gas [48]. The hamiltonian describing such a system is typically of the form

$$H = \int d^3\mathbf{r} \left\{ \psi^\dagger(\mathbf{r}) \left(\frac{\boldsymbol{\nabla}^2}{2m} - \mu \right) \psi(\mathbf{r}) + \frac{g}{2} \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \psi(\mathbf{r}) \right\}, \tag{46}$$

where $g = 4\pi a/m$, with a the s-wave scattering length. (We are ignoring here a subtlety related to the ultraviolet divergences that are generated in calculations with a contact interaction and that require the introduction of an ultraviolet cut-off. This can be implemented in a standard fashion, but it plays no role in the present discussion.) The effective hamiltonian (46) provides an accurate description of dilute systems, when the scattering length is small compared to the interparticle distance, i.e., $an^{1/3} \ll 1$. In the vicinity of the condensation, where $n^{1/3}\lambda \simeq 1$, with $\lambda = \sqrt{\frac{2\pi}{mT}}$ the thermal wavelength, the diluteness condition reads $a/\lambda \ll 1$.

It has been shown that the shift ΔT_c of the Bose-Einstein condensation temperature is linear in $an^{1/3}$ [48]:

$$\frac{\Delta T_c}{T_c^0} = \frac{T_c - T_c^0}{T_c^0} = c\ an^{1/3}. \tag{47}$$

Here T_c^0 is the condensation temperature of the ideal gas, defined by the condition $n\lambda^3 = \zeta(3/2) \approx 2.612$, and T_c the transition temperature of the interacting system at the same density. This result is non trivial: although the shift is proportional to a, and hence is small if a is small, the result (47) cannot be obtained form perturbation theory. It is obtained through the following chain of arguments. In the limit of small coupling the shift of the critical temperature may be obtained from the shift of the critical density, $\frac{\Delta T_c}{T_c^0} = -\frac{2}{3}\frac{\Delta n_c}{n_c^0}$. The latter quantity is easier to calculate, and is dominated by the contribution of the zero Matsubara frequency part, ψ_0, of the bosonic field, whose dynamics is governed by a three-dimensional classical field theory with $O(2)$ symmetry. To make the connection with the notation used in the rest of this section, we set $\psi_0 = \sqrt{mT}(\varphi_1 + i\varphi_2)$, with φ_1 and φ_2 two real fields. The action for these real fields is given by Eq. (41), with the parameters r and u related to the parameters of the hamiltonian (46) by

$$r = -2mT\mu, \qquad u = 96\pi^2 \frac{a}{\lambda^2}. \tag{48}$$

The resulting density shift is then given by the change in the fluctuation of this classical field, with the coefficient c in Eq. (47) given by

$$c = -\frac{256\pi^3}{(\zeta(3/2))^{4/3}} \frac{\Delta\langle\varphi_i^2\rangle}{Nu}, \tag{49}$$

in the limit $u \to 0$ (and for $N = 2$).

The calculation of $\Delta\langle\varphi_i^2\rangle$ is difficult for the reason already mentioned: although the coupling constant u can be arbitrarily small, perturbation theory cannot be used because of the infrared divergences of the critical three-dimensional field theory. The best numerical estimates for $\Delta\langle\varphi^2\rangle$, and hence for c, are those which have been obtained using the lattice technique by two groups, with the results: $c = 1.32 \pm 0.02$ [51] and $c = 1.29 \pm 0.05$ [52]. The availability of these results has turned the calculation of c into a testing ground for other non perturbative methods: expansion in $1/N$ [49, 50], optimized perturbation theory [55, 56], resummed perturbative calculations to high loop orders [53].

To understand better the origin of the difficulty of the calculation of c, as well as the linearity in a of ΔT_c, let us write $\Delta\langle\varphi_i^2\rangle$ as the following integral

$$\frac{\Delta\langle\varphi_i^2\rangle}{N} = \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{p^2+\Sigma(p)} - \frac{1}{p^2}\right) = -\frac{1}{2\pi^2}\int \frac{dp}{p}\left[p - \frac{p^3}{p^2+\Sigma(p)}\right]. \tag{50}$$

where $\Sigma(p)$ is the self-energy at criticality, i.e., $\Sigma(0) = 0$. In eq. (50), the term within the square brackets is, to a very good approximation (when p/u is small enough), equal to $\Sigma(p)/p$, a function that is peaked in the region of intermediate momenta between the critical region and the high momentum perturbative region. Thus, the difficulty in getting a precise evaluation of the integral (50) is that it requires an accurate determination of $\Sigma(p)$ in a large region of momenta including the crossover region between two different physical regimes [49, 57]. In that sense, the calculation of c can be viewed as a very stringent test of the approximation scheme, and in fact, it is in order to obtain a reliable estimate of c that the BMW scheme was initially developed.

To see the origin of the linear relation between Δn_c and a, we note first that the action (1) contains a single dimensionfull parameter, u, r being adjusted for any given u to be at criticality. In fact the effective three dimensional theory is ultraviolet divergent, so there is a priori another parameter, an ultraviolet cut-off $\Lambda \sim 1/\lambda$, with λ the thermal wavelength. It follows then from dimensional analysis that $\Sigma(p)$ can be written as $\Sigma(p = xu) = u^2\sigma(x, u/\Lambda)$. Now, the diagrams involved in the calculation of Σ (at criticality) are ultraviolet convergent, so that the infinite cut-off limit can be taken. Note however that this requires that all

TABLE II: Coefficient c for the $O(N)$ models.

N	BMW	Resummed pert. th.		Monte-Carlo	
	c	c	Ref.	c	Ref.
1	1.15	1.07(10)	[53]	1.09(9)	[54]
2	1.37	1.27(10)	[53]	1.32(2)	[51]
3	1.50	1.43(11)	[53]		
4	1.63	1.54(11)	[53]	1.6(1)	[54]

momenta involved in the various integrations are small in comparison with Λ or, in other words, that the integrands are negligibly small for momenta $k \sim \lambda^{-1}$. Only then can we ignore the effects of non vanishing Matsubara frequencies, and finite cut-off effects. This implies that $u\lambda \sim a/\lambda$ is sufficiently small. In this region of validity of the classical field approximation, that is, for small enough u, $\sigma(x, u/\Lambda)$ becomes a universal function $\sigma(x)$, independent of u, and

$$\frac{\Delta\langle\varphi_i^2\rangle}{Nu} = -\frac{1}{2\pi^2}\int dx \frac{\sigma(x)}{x^2+\sigma(x)}, \tag{51}$$

showing that the change in the critical density is indeed linear in u, and hence in a.

Table II contains our results for c together with some of the best estimates available in the literature. As was the case for the critical exponents in Table I, our numbers are all given for the PMS values α^* of the regulator parameter, and the digits quoted are those which remain stable when α varies in the range $[\alpha^* - \frac{1}{2}, \alpha^* + \frac{1}{2}]$. For all N values where six-loop resummed calculations exist, our results for c are within the error bars. For $N = 100$, we find $c = 2.36$, which compares well to the exact large N value $c \simeq 2.33$ [49]. Our estimates for c are also comparable to those obtained from an approximation specifically designed for this quantity [13, 60]. See also [58] for an earlier estimate of c using a different truncation of the RG equations, and also [59].

C. Thermodynamics of quantum fields

The last application that we shall consider concerns the thermodynamics of quantum fields at high temperature. This is motivated by the study of the quark-gluon plasma. In such a system one could expect a priori perturbation theory to yield accurate results because the asymptotic freedom of Quantum Chromodynamics makes the effective coupling small at

high temperature. However, strict perturbation theory does not work: it exhibits indeed very poor convergence properties, even in a range of values of the coupling constant where reasonable results are obtained at $T = 0$. This difference of behavior of perturbation theory at zero and finite temperature can be understood from the fact that, at finite temperature, the actual expansion parameter involves both the coupling constant and the magnitude of thermal fluctuations (for a recent review, see [61]; see also [62]). In that respect, the problem is not specific to QCD: Similar poor convergence behavior appears also in the simpler scalar field theory [63], and has also been observed in the case of large-N φ^4 theory [64]. We focus here on the case of scalar field with a $g^2\varphi^4$ interaction (i.e., $g^2 \equiv u/24$) [35].

Let us first recall how the effect of the interactions at a given scale depends on the magnitude of the relevant thermal fluctuations at that scale (and in some cases at a different scale as well). The thermal fluctuations of the field are given by the following integral

$$\langle\varphi^2\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{n_k}{k}, \qquad n_k = \frac{1}{\mathrm{e}^{k/T}-1}. \tag{52}$$

When we perform a perturbative calculation, we assume that the "kinetic energy" $\sim \langle(\partial\varphi)^2\rangle$ is large compared to the "potential energy" $\sim g^2\langle\varphi^4\rangle$. Obviously, this comparison depends on the strength of the coupling, but also on the typical wavelength, or momentum, of the fluctuations. To make things more precise, let us observe that the integral (52) is dominated by the largest values of k (in the absence of the statistical factor it would be quadratically divergent). One may then calculate the integral with an upper cut-off κ and refer to the corresponding value as to "the contribution of the fluctuations at scale κ", and denote it by $\langle\varphi^2\rangle_\kappa$. In the same spirit, we may approximate the kinetic energy of modes at scale κ as $\langle(\partial\varphi)^2\rangle_\kappa \approx \kappa^2\langle\varphi^2\rangle_\kappa$. Assuming furthermore that $\langle\varphi^4\rangle_\kappa \approx \langle\varphi^2\rangle_\kappa^2$, one gets the expansion parameter (ratio of potential energy $\sim g^2\langle\varphi^2\rangle_\kappa^2$ to kinetic energy $\sim \kappa^2\langle\varphi^2\rangle_\kappa$)

$$\gamma_\kappa = \frac{g^2\langle\varphi^2\rangle_\kappa}{\kappa^2}. \tag{53}$$

Let us then examine the values of this parameter for several characteristics momenta. The fluctuations that dominate the energy density at weak coupling correspond to the plasma particles and have momenta $k \sim T$. For these "hard" fluctuations,

$$\kappa \sim T, \qquad \langle\varphi^2\rangle_T \sim T^2, \qquad \gamma_T \sim g^2. \tag{54}$$

Thus, at this scale, perturbation theory works as well as at zero temperature (with expansion parameter $\sim g^2$, or rather $\alpha = g^2/4\pi$).

The next "natural" scale, commonly referred to as the "soft scale", corresponds to $\kappa \sim gT$. We have then

$$\kappa \sim gT, \qquad \langle\varphi^2\rangle_{gT} \sim gT^2, \qquad \gamma_{gT} \sim g. \tag{55}$$

(In calculating $\langle\varphi^2\rangle_\kappa$ for $\kappa \ll T$, we have used the approximation $n_k \approx T/k$, so that $\langle\varphi^2\rangle_{\kappa\ll T} \sim \kappa T$.) Since $\gamma_{gT} \sim g$, perturbation theory can still be used to describe the self-interactions of the soft modes. However the perturbation theory is now an expansion in powers of g rather than g^2: it is therefore less rapidly convergent. The emergence of this new expansion parameter is the origin of odd powers of g in the perturbative expansion of the pressure (such as the plasmon term $\sim g^3$).

Another phenomenon occurs at the scale gT. While the expansion parameter $\gamma_{gT} \sim g$ that controls the self-interactions of the soft fluctuations is small, the coupling between the soft modes and the thermal fluctuations at scale T is not: indeed $g^2\langle\varphi^2\rangle_T \sim (gT)^2$, that is, the kinetic energy of the soft modes $\sim (gT)^2$ is comparable to their interaction energy resulting from their coupling to the hard modes, $\sim g^2\langle\varphi^2\rangle_T$. Thus the dynamics of soft modes is non-perturbatively renormalized by their coupling to hard modes. This particular coupling is encompassed by the so-called "hard thermal loops" [66].

Finally, there is yet another scale, the "ultra-soft scale" $\kappa \sim g^2T$, at which perturbation theory completely breaks down. At this scale, we have indeed

$$\kappa \sim g^2T, \qquad \langle\varphi^2\rangle_{gT} \sim g^2T^2, \qquad \gamma_{g^2T} \sim 1. \tag{56}$$

Thus the ultra-soft fluctuations remain strongly coupled for arbitrarily small values of the coupling constant. This situation does not occur for a scalar field since a mass is generated at scale gT, which renders the contribution of the fluctuations at the scale g^2T negligible. However this situation is met in QCD for the long wavelength, unscreened, magnetic fluctuations.

These considerations suggest that the main difficulty with thermal perturbation theory is not so much related to the magnitude of the coupling constant (for the relevant temperatures it is not that large), but it is rather due to the interplay of degrees of freedom with various wavelengths, possibly involving collective modes. In some sense, field theories at high temperature are multiscale systems. At weak coupling the dynamically generated scales T, gT and g^2T are well separated. This allows for instance the organization of the calculation using effective field theory. However the scale separation disappears when the coupling is not too

small: then, the various degrees of freedom mix and the situation requires a different type of analysis.

The exact renormalization group is ideally suited to cope with this type of situations. In particular, as we have seen, the BMW truncation scheme provides an excellent description of the momentum dependence of the 2-point function, from the low momenta of the critical region, all the way up to the large momenta of the perturbative regime. One may then expect this method to capture accurately the contributions to the thermodynamical functions of thermal fluctuations from various momentum ranges, and hence handle properly the mixing between degrees of freedom that takes place as the coupling grows. Since it involves also non trivial momentum dependent self-energies, the method also encompasses implicitly effects related to the damping of quasiparticles, or their coupling to complex multi-particle configurations.

Fig. 3 displays the pressure as the function of the coupling constant at the scale $g(2\pi T)$. The various diverging curves labelled $g^2, \cdots, g^8 \ln g$ indicate the results of perturbative calculations, up to order $g^8 \ln g$ (for recent high order calculations of the thermodynamics of the scalar field, see [65]). These curves clearly illustrate the poor behavior of strict perturbation theory. The other curves correspond to various implementations of the exact renormalization group, as well as to a 2PI calculation based on a simple 2-loop skeleton [34]. The two curves labelled LPA correspond to two different choices of regulators: either the Litim regulator [68], which is implemented only for three-momenta, with the (untruncated) sums over the Matsubara frequencies being performed analytically) [34], and an exponential regulator that affects both the momenta and the frequencies. The BMW approximation scheme is better justified when one uses an Euclidean symmetric four dimensional regulator, and the calculations reported here have been done with the exponential regulator (42).

In contrast to the perturbative calculation, the calculations based on the renormalization group show a remarkable stability, and a smooth extrapolation towards strong coupling. As it turns out, the results obtained are not too different from those of the LPA, nor from the simple 2-loop 2PI approximation used in Ref. [34]. In physical terms, both the LPA and the 2-loop 2PI approximation correspond to approximations where the degrees of freedom of the hot scalar plasma are massive quasiparticles. The new scheme goes beyond that simple picture. This stability of the results against improvements in the approximation suggests that the scheme that we are using to solve the NPRG equations may give already, at the level at which it is implemented here, an accurate representation of the exact pressure, and this over

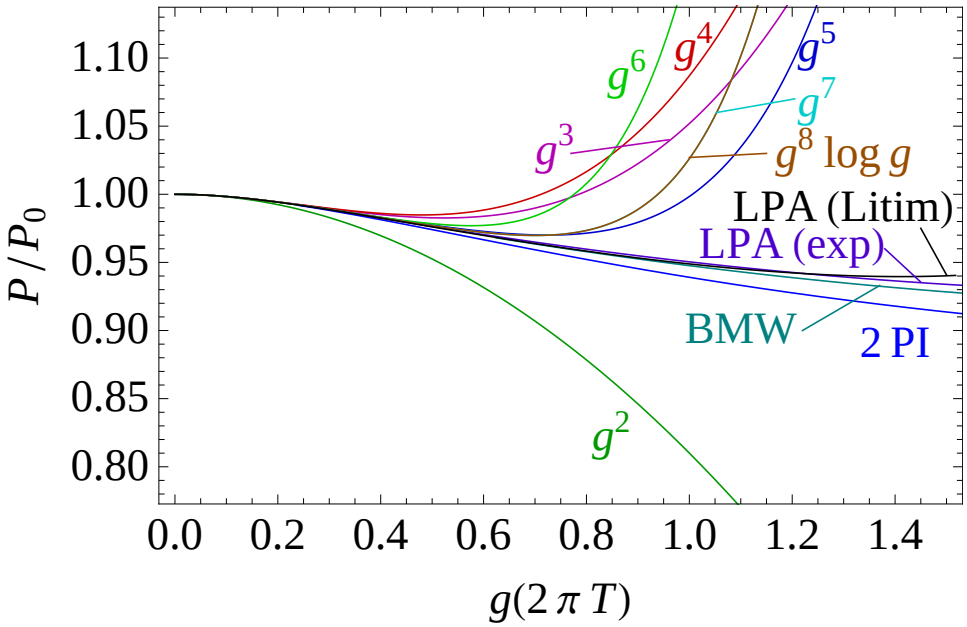

FIG. 3: (*Color online*) Pressure as function of the coupling. The various resummation and RG methods compared are 2 PI, LPA with exponential regulator (exp), LPA with Litim regulator, and BMW. Shown are also perturbative results through order $g^6 \log g$ for the mass and $g^8 \log g$ for the pressure. The g^7 and $g^8 \log g$ curves for the pressure almost lie on top of each other, as do the BMW and LPA (exp) curves for the mass.. From Ref. [35]

a wide range of coupling constants. It also indicates that for such a system the quasiparticle picture is presumably robust.

V. CONCLUSION

The few applications that are presented in the previous section illustrate the power of the renormalization group, when coupled to an approximation scheme that allows for a determination of the full momentum dependence of the n-point functions. The equations that need to be solved are a priori complicated: these are flow equations which are at the same time partial differential equations (with partial derivatives with respect to the background field), with integral kernels that involve the solution itself. Still they can be solved at a rather modest numerical cost, using elementary numerical techniques. The studies presented here involved only the leading-order of a systematic approximation scheme. In the absence of any small parameter controlling the magnitude of successive orders, a study of the next order would be necessary in order to quantify the accuracy that has been

reached. However, the robustness of the leading order results can already be gauged from their weak residual dependence on the choice of the regulator.

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